

FRACTIONAL BROWNIAN FIELDS OVER MANIFOLDS

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ABSTRACT. Extensions of the fractional Brownian fields are constructed over a complete Riemannian manifold. This construction is carried out for the full range of the Hurst parameter $\alpha \in (0, 1)$. In particular, we establish existence, distributional scaling (self-similarity), stationarity of the increments, and almost sure Hölder continuity of sample paths. Stationary counterparts to these fields are also constructed.

1. INTRODUCTION

The fractional Brownian motions and their stationary counterparts are the basic examples of Gaussian random fields over \mathbb{R} and it is natural to ask what are the corresponding examples when \mathbb{R} is replaced by a manifold. The first to do so was Paul Lévy (see [25]), who extended the standard Brownian motion on \mathbb{R} to the standard Brownian field over \mathbb{R}^d , now called Lévy's Brownian motion. Lévy then extended this field to the sphere \mathbb{S}^d . Since then there have been a number of studies aimed at extending both the Brownian motion and the fractional Brownian motion to other manifolds. This is a natural step in the theory of Gaussian fields in general as one would like to understand how the structure of the index set determines the kinds of fields that can be defined over it. The geometric and topological structure of Riemannian manifolds make them a convenient and interesting setting for such a study. When one extends the fractional Brownian motions from \mathbb{R} to \mathbb{R}^d the resulting fields are called *fractional Brownian fields* (some authors prefer *Levy fractional Brownian motions*) and our purpose in this article is to construct fields over Riemannian manifolds that generalize the fractional Brownian fields over \mathbb{R}^d .

Much of the interest in the fractional Brownian fields (fBf 's) over \mathbb{R}^d stems from their distributional invariance and scaling properties. In particular, if $\alpha \in (0, 1)$ denotes the Hurst index and the corresponding field is denoted by fBf^α , the increments of the fBf^α are invariant under rotation and translation and the distribution of the fBf^α scales by a power c^α when \mathbb{R}^d is dilated by $c > 0$. Any extension of the fBf 's should possess these properties and also reflect the geometry of the index set in question (for an introduction to Gaussian random fields over manifolds focusing on smooth fields see the recent work of Adler and Taylor [2] and the lecture notes from the short course given at the 2012 joint meetings of the AMS¹).

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¹<http://www.ams.org/meetings/short-courses/short-course-general#random%20fields>

As mentioned above the first attempt to extend Lèvy's Brownian motion, $fBf^{\frac{1}{2}}$, from \mathbb{R}^d to a manifold was by Lèvy himself in [25]. There he constructed a field over \mathbb{S}^d with covariance given by

$$d(x, o) + d(y, o) - d(x, y),$$

$d(x, y)$ being the geodesic distance between x and y and o being a fixed origin point on the sphere. Further progress in this direction was made in the work of Molchan (see e.g. [28]) and Gangolli (see [16]) where the authors dealt with extensions of Lèvy's Brownian motion to other manifolds including the sphere.

Most recently Istas in [21] studied fields over certain Riemannian manifolds with covariance given by

$$(1.1) \quad d(x, o)^{2\alpha} + d(y, o)^{2\alpha} - d(x, y)^{2\alpha}$$

where $d(x, y)$ is the metric of the manifold and o is a chosen point. In particular Istas showed there that (1.1) defines a Gaussian field over compact rank one symmetric spaces and hyperbolic space \mathbb{H}^d if and only if $\alpha \in (0, 1/2]$.

A common feature of the above approaches is that they begin by looking for covariances of the form $f(x, o) + f(y, o) - f(x, y)$ for some symmetric function f ; the idea being that over \mathbb{R}^d $o = 0$ and $f(x, y) = \|x - y\|_{\mathbb{R}^d}$. The issue then is to prove that the function so defined does, in fact, define a covariance, i.e., one must establish positive definiteness. A necessary and sufficient condition for positive definiteness is that f be of *negative type*, for example one can take the above approach on metric spaces (X, d) with metric of negative type (e.g. [22, 20]). In general if $d(x, y)$ is the metric of a Riemannian manifold, establishing that $d(x, y)^{2\alpha}$ is of negative type for some $\alpha \in (0, 1)$ is non-trivial and indeed, as in [21], it has been shown $d(x, y)^{2\alpha}$ can fail to be of negative type. Moreover, in all the above work this approach necessitates symmetry assumptions on the underlying manifold.

In the present article we take an essentially different approach inspired by the work of Benassi, Jaffard, and Roux (see [4] and more recently [5]). In particular we extend a characterization of the fBf^α in terms of the Laplacian on \mathbb{R}^d to the Riemannian setting via the Laplace-Beltrami operator and the associated heat kernel. Using this approach we are able to extend the fBf^α to a variety of both compact and non-compact manifolds without any assumptions regarding symmetry of the manifolds and for the full range of $\alpha \in (0, 1)$ (see Theorems 3.2-3.4 below).

This article is structured as follows: in Section 2 we cover some preliminaries regarding Gaussian random fields and analysis on manifolds, in particular the heat kernel of a Riemannian manifold. In Section 3.1 we describe the motivation behind our approach before moving on to discuss existence in Section 3.2. In Sections 3.3 and 3.4 we establish distributional properties and sample path regularity. In Section 4 we construct stationary counterparts the fields of Section 3 and establish the corresponding distributional properties. Section 5 contains some open questions concerning geometry and probability encountered in the course of the article and in the appendix we collect some auxiliary results concerning sample path regularity of Gaussian fields over manifolds.

2. PRELIMINARIES

2.1. Gaussian Random Fields. Given a complete probability space (Ω, \mathcal{F}, P) and some index set I we call a collection of random variables on Ω , $\{X_i(\omega)\}_{i \in I}$, a *Gaussian random field (GRF) over I* if for any finite subset $\{i_k\}_1^n \subset I$ the random vector $(X_{i_k})_1^n$ has a joint normal distribution. Then for each $\omega \in \Omega$, $X_i(\omega)$ defines a real valued function on I called a *sample path* of the field $\{X_i\}$. We let \mathbb{E} denote the expectation operator,

$$\mathbb{E}[X_i] \equiv \int_{\Omega} X_i(\omega) dP(\omega) \quad i \in I$$

and we call

$$\mathbb{E}[(X_s - \mathbb{E}[X_s])(X_t - \mathbb{E}[X_t])] = \mathbb{E}[X_s X_t] - \mathbb{E}[X_s]\mathbb{E}[X_t] \quad s, t \in I$$

the *covariance* of $\{X_i\}$. The covariance of a GRF over I defines a symmetric positive definite function on $I \times I$.

We say two GRF's are equal in *finite dimensional distribution* or simply *in distribution*, denoted $\stackrel{d}{=}$, if their covariances are equal. We also say two GRF's are *versions* of each other if they are equal in distribution. The salient analytical feature of GRF's is that for any set I the collection of all GRF's over I is in one to one correspondence up to equality in distribution with the set of all symmetric, positive definite functions on $I \times I$. In other words a GRF is uniquely determined in distribution by its covariance and every symmetric positive definite function on $I \times I$ is the covariance of a GRF over I .

We call a GRF *centered* if $\mathbb{E}[X_i] = 0 \forall i \in I$ and in this case its covariance is given by $\mathbb{E}[X_t X_s]$, $s, t \in I$. Throughout this article we will only consider centered GRF's.

2.1.1. The Fractional Brownian Fields Over \mathbb{R}^d . The standard Brownian motion B_t over $[0, \infty)$ is the centered GRF with covariance

$$\mathbb{E}[B_s B_t] = s \wedge t = \frac{|s| + |t| - |t - s|}{2}.$$

From this one generalizes to obtain the fractional Brownian motion fBm^α for $\alpha \in (0, 1)$:

$$\mathbb{E}[fBm_s^\alpha fBm_t^\alpha] = \frac{|s|^{2\alpha} + |t|^{2\alpha} - |t - s|^{2\alpha}}{2}.$$

We then have $B_t = fBm_t^{\frac{1}{2}}$.

One then further generalizes to \mathbb{R}^d , obtaining the fBf^α as the centered GRF over \mathbb{R}^d with covariance

$$\mathbb{E}[fBf_x^\alpha fBf_y^\alpha] = \|x\|_{\mathbb{R}^d}^{2\alpha} + \|y\|_{\mathbb{R}^d}^{2\alpha} - \|x - y\|_{\mathbb{R}^d}^{2\alpha}$$

(note that some authors include the constant factor $1/2$).

One easily sees that the fBf^α is *self similar* of order α , i.e., if fBf_c^α denotes the field rescaled field $\{fBf_{cx}^\alpha\}_{x \in \mathbb{R}^d}$ then

$$fBf_c^\alpha \stackrel{d}{=} c^\alpha fBf^\alpha \quad \forall c > 0,$$

and that it has *stationary (or homogeneous) increments*:

$$\mathbb{E}[|fBf_x^\alpha - fBf_y^\alpha|^2] = \|x - y\|^{2\alpha} = \|\iota(x) - \iota(y)\|^{2\alpha} = \mathbb{E}[|fBf_{\iota(x)}^\alpha - fBf_{\iota(y)}^\alpha|^2]$$

for any isometry ι on \mathbb{R}^d . Moreover it is known that there exists a version X_x of the fBf^α such that with probability one the sample paths $X_x(\omega)$ are Hölder continuous of any order $\gamma < \alpha$ and fail to be Hölder continuous of any order $\gamma > \alpha$ at every point in \mathbb{R}^d (see [1]).

2.1.2. White Noise. The treatment here follows [23]. Given a probability space (Ω, \mathcal{F}, P) we call a complete subspace G of $L^2(\Omega, \mathcal{F}, P)$ a *Gaussian Hilbert space* if every element of G is a centered Gaussian random variable. Note that the inner product H inherits from $L^2(\Omega, \mathcal{F}, P)$ is then

$$\langle X, Y \rangle_G = \mathbb{E}[XY].$$

Given any (real) Hilbert space H there exists a Gaussian Hilbert space G and a unitary map $W : H \rightarrow G$ called W the *isonormal process* or *white noise process* on H (one can also consider complex white noises, but we will only consider the real case). If, as is the case below, $H = L^2(M, \mathcal{S}, d\mu)$ for some measure space $(M, \mathcal{S}, d\mu)$ then if $B = \{A \in \mathcal{S} : \mu(A) < \infty\}$ the map from $B \rightarrow G$ given by

$$W(A) \equiv W(\chi_A)$$

determines a Gaussian random measure on M . The properties of such measures will not be important for us here, but we mention them to motivate the notation for $W : H \rightarrow G$, given by

$$W(f) = \int_M f(z) dW(z),$$

which we refer to as a *white noise integral* (this is also commonly called a *stochastic integral*). Starting from a random measure one can construct the integral $\int_M dW$ in close analogy with classical measure theory. All that will be important for us is the property

$$\langle f, g \rangle_H = \mathbb{E} \left[\int_M f dW, \int_M g dW \right].$$

Now suppose we have a function $h(x, z) : M \rightarrow L^2(M, d\mu)$, $x \mapsto h(x, z) \in L^2(M, d\mu(z))$. We can then define a centered GRF Y_x over X by

$$Y_x \stackrel{d}{=} \int_M h(x, z) dW(z).$$

The covariance of Y_x is then given by

$$\mathbb{E}[Y_x Y_y] = \langle h(x, z), h(y, z) \rangle_{L^2} = \int_M h(x, z) h(y, z) d\mu(z).$$

Note that the last expression on the right is in fact positive definite and symmetric. In this case we call h the integral kernel of Y .

2.2. Analysis on Manifolds. In what follows we assume throughout that all Riemannian manifolds are complete and of dimension d , with $2 \leq d < \infty$. For a manifold M let Δ denote the Laplace-Beltrami operator, or simply the Laplacian for short, on M . In any local coordinate system the action of Δ on $C^\infty(M)$ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum \partial_j (g^{ij} \sqrt{g} \partial_i)$$

where (g_{ij}) is the matrix of the Riemannian metric in these coordinates, $(g^{ij}) = (g_{ij})^{-1}$, and $\sqrt{g} = (\det(g_{ij}))^{\frac{1}{2}}$. Because M is complete, Δ is essentially self adjoint (see e.g. [30]) and so we may consider from now on the unique minimal self-adjoint extension of Δ , which we shall write as Δ also. Moreover the spectrum of Δ is contained in $(-\infty, 0]$ (see e.g. [30]). By the spectral theorem we can define the heat semigroup

$$e^{t\Delta} = \int_0^\infty e^{-t\lambda} dE_\lambda$$

where dE_λ is the spectral measure of $-\Delta$. The action of $e^{t\Delta}$ on $L^2(M, dV_g)$, where dV_g denotes the measure derived from the metric g , is given by a kernel $H_t(x, y)$:

$$e^{t\Delta}(f)(x) = \int_M H_t(x, y) f(y) dV_g(y).$$

$H_t(x, y)$ is called the *heat kernel* of M . It is known that H_t is strictly positive, symmetric, and contained in $C^\infty(M \times M \times (0, \infty))$. Moreover we have the semigroup property

$$\int_M H_t(x, z) H_s(z, y) dV_g(z) = H_{t+s}(x, y).$$

As a consequence H_t is positive definite for each $t > 0$. As its name suggests, $H_t(x, y)$ is a fundamental solution to the heat equation on $M \times (0, \infty)$:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta_x \right) H_t(x, y) = 0 \\ \lim_{t \downarrow 0} \int_M H_t(x, y) f(y) dV_g(y) = f(x) \quad \forall f \in C_0(M). \end{cases}$$

There are various constructions of the heat kernel, that given in [10] being most suited to our purposes. In particular if we let

$$\mathcal{E}_t(x, y) \equiv \frac{e^{-\frac{d(x, y)^2}{4t}}}{\sqrt{(4\pi t)^d}}$$

then there is an open neighborhood of the diagonal $U \subset M \times M$ such that on U

$$(2.1) \quad \frac{H_t(x, y)}{\mathcal{E}_t(x, y)} = \Phi(t, x, y)$$

where $\Phi(t, x, y)$ is symmetric in x and y , $\Phi \in C^k([0, T] \times U) \forall T > 0$ where k can be chosen arbitrarily large (see [7] and [6]), and

$$\lim_{t \rightarrow 0, x \rightarrow y} \Phi(t, x, y) = 1.$$

In other words, for x and y close $H_t \sim \mathcal{E}_t$ as $t \rightarrow 0$. Thus on any manifold heat diffusion behaves locally for small times as in Euclidean space.

If M is compact then we also have the following eigenfunction expansion of H_t :

$$(2.2) \quad H_t(x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

where $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_k \uparrow \infty$ and $\{\phi_k\}$ are the spectrum and orthonormalized L^2 eigenfunctions of $-\Delta$ respectively and where (2.2) converges absolutely and uniformly for each $t > 0$ (see [7]).

Following [7] we define a *regular domain* to be an open, connected, relatively compact subset D of a complete Riemannian manifold such that $\partial D \neq \emptyset$ is smooth. In what follows when we refer to the Laplacian of a regular domain we mean the Dirichlet Laplacian with corresponding the heat kernel (see [7], Chapter 7). As in the compact case we have an eigenfunction expansion (2.2), the only difference being that $\lambda_0 > 0$. If (M, g) is a regular domain in manifold (N, g) then, as noted in [8], (2.1) holds in this setting as well.

Now suppose M is complete and non-compact, $\{D_k\}_1^\infty$ is any increasing exhaustion of M by regular domains, and $H_t^k(x, y)$ denotes the Dirichlet heat kernel of D_k . Then if we extend each H^k to be zero outside $\overline{D} \times \overline{D}$, $\{H_t^k(x, y)\}_1^\infty$ forms a pointwise increasing sequence on $M \times M \times (0, \infty)$. It was shown in [14] that

$$\lim_{k \rightarrow \infty} H_t^k(x, y) = H_t(x, y)$$

where $H_t(x, y)$ is the heat kernel defined above.

3. THE RIESZ FIELDS

3.1. Motivation and Definition. In [4] the authors begin by defining a symbol class of pseudodifferential operators over \mathbb{R}^d . From such an operator A they define a Gaussian random field with covariance given by the integral kernel of A^{-1} . The authors are then able to derive all the important properties of this field from properties of the symbol of the operator A . This approach to constructing and studying GRF's is a natural extension of the classical spectral theory of Gaussian processes on \mathbb{R} and provides a beautiful demonstration of the power of the spectral point of view.

The basic heuristic can be described as follows: Beginning with an unbounded operator A on some L^2 space, define and study the GRF determined by the integral kernel of A^{-1} . So in attempting to extend the fBf^α to a Riemannian manifold, we should first seek an operator A that determines the fBf^α in the manner above.

Our starting point is the well known (e.g. [4] or [32]) spectral representation of the fBf^α ,

$$(3.1) \quad fBf_x^\alpha \stackrel{d}{=} C_{d,\alpha} \int_{\mathbb{R}^d} \frac{e^{i\langle x, \xi \rangle} - 1}{\|\xi\|^{\frac{d}{2} + \alpha}} d\widehat{W}(\xi),$$

where \widehat{W} is a complex white noise on $L^2(\mathbb{R}^d, dx)$, dx is Lebesgue measure, and $C_{d,\alpha}$ is a constant. Examining (3.1) we see that, up to a constant, for $f \in H_{-(\frac{d}{4} + \frac{\alpha}{2})}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \frac{e^{i\langle x, \xi \rangle} - 1}{\|\xi\|^{\frac{d}{2} + \alpha}} \hat{f}(\xi) d\xi = (-\Delta)^{-(\frac{d}{4} + \frac{\alpha}{2})}(f)(x) - (-\Delta)^{-(\frac{d}{4} + \frac{\alpha}{2})}(f)(0).$$

Thus if we denote this last operator above by A then the fBf^α is the unique (in distribution) GRF with covariance given by the Schwarz kernel of the operator A^*A ,

$$\mathbb{E}[fBf_x^\alpha fBf_y^\alpha] = C \int_{\mathbb{R}^d} \frac{e^{i\langle x-y, \xi \rangle} - e^{i\langle x, \xi \rangle} - e^{i\langle y, \xi \rangle} + 1}{\|\xi\|^{d+2\alpha}} d\xi.$$

We now have a characterization that extends immediately to manifolds: Simply replace the Laplacian on \mathbb{R}^d by the Laplace-Beltrami operator of the manifold in question and determine the kernel of the operator A^*A . Following [30] we arrive at the following definitions:

Definition 3.1. For a complete Riemannian manifold M with heat kernel $H_t(x, y)$ define the *Riesz field* R^α to be the GRF with covariance given by

$$(3.2) \quad \mathbb{E}[R_x^\alpha R_y^\alpha] \equiv \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} (H_t(x, y) - H_t(x, o) - H_t(y, o) + H_t(o, o)) dt$$

where $o \in M$ is a fixed “origin” and the *stationary (or homogeneous) Riesz field* hR^α the GRF with covariance

$$(3.3) \quad \mathbb{E}[hR_x^\alpha hR_y^\alpha] \equiv \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} H_t(x, y) dt.$$

Because $H_t(x, y)$ is positive definite for each $t > 0$ and

$$\begin{aligned} & H_t(x, y) - H_t(x, o) - H_t(y, o) + H_t(o, o) \\ &= \int_M (H_{t/2}(x, z) - H_{t/2}(o, z)) (H_{t/2}(y, z) - H_{t/2}(o, z)) dV_g(z), \end{aligned}$$

each of these expressions is symmetric and positive definite, and thus when the integrals exist each determines a GRF over M . Of course the convergence of the above integrals is by no means obvious and our first task in Section 3.2 will be to determine manifolds for which they do converge.

Remark 3.1. We will see shortly that if either (3.2) or (3.3) exist for some $\alpha_0 \in (0, 1)$ then it also exists for any $\alpha \in (0, \alpha_0)$. We say R^α (resp. hR^α) *exists* for all $\alpha \in (0, b)$ if (3.2) (resp. (3.3)) is finite for all $\alpha \in (0, b)$, $b \leq 1$, and all $x, y \in M$.

It turns out (Proposition 3.5) that the Riesz field (3.2) extends the fBf^α and that they agree up to a constant in distribution over \mathbb{R}^d . However we will also see that the stationary Riesz field has some claim to be an extension of the fBf^α , for example over negatively curved manifolds, even though it does not exist on \mathbb{R}^d .

Now let W denote the white noise over $L^2(M, dV_g)$. We will show that when they exist the Riesz fields admit the following integral representations:

$$(3.4) \quad R_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} (H_t(x, z) - H_t(o, z)) dt dW(z)$$

and

$$(3.5) \quad hR_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x, z) dt dW(z).$$

The issue is whether or not the functions appearing in the above are in fact square integrable for each $x \in M$. Let us consider this in detail, first for hR^α :

Letting $h_{hR}(x, z) = \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x, z) dt$ we have

$$\begin{aligned} & \langle h_{hR}(x, z), h_{hR}(y, z) \rangle_{L^2} \\ &= \int_M \left(\frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x, z) dt \right) \\ & \quad \times \left(\frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_0^\infty s^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_s(y, z) ds \right) dV_g(z) \\ &= \int_M \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)^2} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x, z) H_s(y, z) dt ds dV_g(z) \\ &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)^2} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} \int_M H_t(x, z) H_s(y, z) dV_g(z) dt ds \\ &= \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)^2} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_{t+s}(x, y) dt ds \\ &= \int_0^\infty \int_s^\infty \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)^2} (t - s)^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} H_t(x, y) dt ds \\ &= \int_0^\infty \int_0^t \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)^2} (t - s)^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} ds H_t(x, y) dt \end{aligned}$$

by the positivity of $H_t(x, y)$ and the semigroup property.

Next note that if $g(s) = \frac{1}{\Gamma(\frac{d}{4} + \frac{\alpha}{2})} s^{\frac{d}{4} + \frac{\alpha}{2} - 1}$ then

$$\int_0^t \frac{1}{\Gamma(\frac{d}{4} + \frac{\alpha}{2})^2} (t-s)^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} ds = g * g(t)$$

where $*$ denotes the finite convolution $f * g(t) \equiv \int_0^t f(t-s)g(s) ds$. If \mathcal{L} denotes the Laplace transform we have the well known property $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$. Applying this to $g * g$ above we have

$$\mathcal{L}(g * g)(s) = (\mathcal{L}(g))^2(s) = \left(s^{-(\frac{d}{4} + \frac{\alpha}{2})}\right)^2 = s^{-(\frac{d}{2} + \alpha)}.$$

Then inverting \mathcal{L} we obtain

$$\frac{1}{\Gamma(\frac{d}{2} + \alpha)} t^{\frac{d}{2} + \alpha - 1} = \mathcal{L}^{-1}\left(s^{-(\frac{d}{2} + \alpha)}\right) = \int_0^t \frac{1}{\Gamma(\frac{d}{4} + \frac{\alpha}{2})^2} (t-s)^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} ds.$$

Substituting this into the integral defining $\langle h_{hR}(x, z), h_{hR}(y, z) \rangle_{L^2}$ above yields

$$\frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} H_t(x, y) dt.$$

Thus whenever hR^α exists it is given by (3.5).

Turning now to (3.2), let $h_R(x, z) = \frac{1}{\Gamma(\frac{d}{4} + \frac{\alpha}{2})} \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} (H_t(x, z) - H_t(o, z)) dt$.

Then

$$\begin{aligned} & \|h_R(x, z)\|_{L^2}^2 \\ & \leq \int_M \int_0^\infty \int_0^\infty s^{\frac{d}{4} + \frac{\alpha}{2} - 1} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} |H_t(x, z) - H_t(o, z)| |H_s(x, z) - H_s(o, z)| ds dt dV_g(z) \\ & = \int_0^\infty \int_0^\infty s^{\frac{d}{4} + \frac{\alpha}{2} - 1} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \int_M |H_t(x, z) - H_t(o, z)| |H_s(x, z) - H_s(o, z)| dV_g(z) ds dt \\ & \leq \int_0^\infty \int_0^\infty s^{\frac{d}{4} + \frac{\alpha}{2} - 1} t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \|H_t(x, \cdot) - H_t(o, \cdot)\|_2 \|H_s(x, \cdot) - H_s(o, \cdot)\|_2 ds dt \\ & = \left(\int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \|H_t(x, \cdot) - H_t(o, \cdot)\|_2 dt \right)^2 \\ & = \left(\int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \sqrt{H_t(x, x) - 2H_t(x, o) + H_t(o, o)} dt \right)^2. \end{aligned}$$

Recall that if M is any Riemannian manifold then from (2.1) for any $x, y \in M$ we have that $H_t(x, y) = O(t^{-\frac{d}{2}})$ as $t \rightarrow 0$. So then

$$\int_0^1 t^{\frac{d}{4} + \frac{\alpha}{2} - 1} \sqrt{H_t(x, x) - 2H_t(x, o) + H_t(o, o)} dt < \infty$$

and

$$\int_0^1 t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, o) + H_t(o, o)) dt < \infty$$

for all $\alpha \in (0, 1)$.

Next notice that if $\alpha + \epsilon < b$

$$\begin{aligned} & \int_1^\infty t^{\frac{d}{4}+\frac{\alpha}{2}-1} \sqrt{H_t(x, x) - 2H_t(x, o) + H_t(o, o)} dt \\ &= \int_1^\infty t^{\frac{d}{4}+\frac{\alpha}{2}+\epsilon-(1+\epsilon)} \sqrt{H_t(x, x) - 2H_t(x, o) + H_t(o, o)} dt \\ &\leq \left(\int_1^\infty t^{-(1+\epsilon)} dt \right)^{\frac{1}{2}} \left(\int_1^\infty t^{\frac{d}{2}+\alpha+\epsilon-1} (H_t(x, x) - 2H_t(x, o) + H_t(o, o)) dt \right)^{\frac{1}{2}} \end{aligned}$$

by Cauchy-Schwarz. Thus if R^α exists for all $\alpha \in (0, b)$ we may interchange the order of integration as with hR^α to obtain

$$\begin{aligned} & \langle h_R(x, z), h_R(y, z) \rangle_{L^2} \\ &= \frac{1}{\Gamma\left(\frac{d}{2} + \alpha\right)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} (H_t(x, y) - H_t(x, o) - H_t(y, o) + H_t(o, o)) dt \\ &= \mathbb{E}[R_x^\alpha R_y^\alpha] \end{aligned}$$

for all such α .

In either case of (3.2) or (3.3) we see that the integrands are continuous on $(0, \infty)$ so by (2.1) convergence depends only on the behavior of the integrand at infinity. Thus the existence of both R_x^α and hR_x^α will depend on the large-time asymptotics of $H_t(x, y)$. These depend on the manifold in question and we will treat distinct cases below.

3.2. Existence.

3.2.1. *The Compact Case.* We have the following:

Theorem 3.1. *If M is a compact Riemannian manifold, then the Riesz field of order α exists over M for any $\alpha \in (0, 1)$ and the stationary Riesz field does not exist over M for any $\alpha \in (0, 1)$.*

Proof. Recall (2.2):

$$H_t(x, y) = \sum_{k=0}^\infty e^{-\lambda_k t} \phi_k(x) \phi_k(y).$$

We have

$$H_t(x, x) - 2H_t(o, x) + H_t(o, o) = \sum_{k=1}^\infty e^{-\lambda_k t} |\phi_k(x) - \phi_k(o)|^2 = O(e^{-\lambda_1 t}) \quad \forall x \in M$$

and $\lambda_1 > 0$. Then (3.2) is clearly finite for any $x \in M$ and all $\alpha \in (0, 1)$.

To see that hR_x^α does not exist on M notice that $\lim_{t \rightarrow 0} H_t(x, y) = \text{Vol}(M)^{-1} \neq 0$ $\forall x, y \in M$.

□

Theorem 3.2. *If M is regular domain then hR^α , and thus by linearity R^α , exists for any $\alpha \in (0, 1)$.*

Proof. As above let

$$H_t(x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y).$$

Then $\lambda_0 > 0$ and $H_t(x, y) = O(e^{-\lambda_0 t})$ for each $x, y \in M$.

□

We note here that in either case above we may integrate term by term using the eigenfunction expansions of H_t to obtain a series expression for the covariance of R^α and hR^α as follows:

For R^α and M compact we have

$$\begin{aligned} \mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} H_t(x, y) - H_t(x, o) - H_t(y, o) + H_t(o, o) dt \\ &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} \sum_{k=0}^{\infty} e^{-\lambda_k t} (\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o)) dt \\ &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} \sum_{k=1}^{\infty} e^{-\lambda_k t} (\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o)) dt \\ &\leq \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \left(\int_0^\infty t^{\frac{d}{2} + \alpha - 1} \sum_{k=1}^{\infty} e^{-\lambda_k t} |\phi_k(x) - \phi_k(o)|^2 dt \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^\infty t^{\frac{d}{2} + \alpha - 1} \sum_{k=1}^{\infty} e^{-\lambda_k t} |\phi_k(y) - \phi_k(o)|^2 dt \right)^{\frac{1}{2}} \\ &= (\mathbb{E}[|R_x^\alpha|^2] \mathbb{E}[|R_y^\alpha|^2])^{\frac{1}{2}}, \end{aligned}$$

which we know from above to be finite.

Then by dominated convergence we may integrate term by term to obtain

$$\begin{aligned} \mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(\frac{d}{2} + \alpha)}{\lambda_k^{\frac{d}{2} + \alpha}} (\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o)) \\ &= \sum_{k=1}^{\infty} (\lambda_k)^{-(\frac{d}{2} + \alpha)} (\phi_k(x) - \phi_k(o))(\phi_k(y) - \phi_k(o)). \end{aligned}$$

The same equality holds for M a regular domain if we number the spectrum as $\{\lambda_k\}_1^\infty$. Similar arguments show that for M a regular domain

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] = \sum_{k=1}^{\infty} (\lambda_k)^{-\left(\frac{d}{2}+\alpha\right)} \phi_k(x) \phi_k(y).$$

Example 3.1. Let $M = \mathbb{S}^2$. Then in terms of the spherical harmonics $\{Y_{km}\}$ we have

$$H_t(x, y) = \sum_{k=0}^{\infty} e^{-k(k+1)t} \sum_{m=-k}^k Y_{km}(x) Y_{km}(y).$$

Applying the harmonic addition formula we have

$$H_t(x, y) = \sum_{k=0}^{\infty} e^{-k(k+1)t} \frac{2k+1}{4\pi} P_k(\cos \theta_{xy})$$

where P_k is the k -th Legendre Polynomial and $\langle x, y \rangle = \cos \theta_{xy}$. Fixing an origin point $o \in \mathbb{S}^2$ we then have

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = \sum_{k=1}^{\infty} (k(k+1))^{-\left(\frac{d}{2}+\alpha\right)} \frac{2k+1}{4\pi} (P_k(\cos \theta_{xy}) - P_k(\cos \theta_{xo}) - P_k(\cos \theta_{yo}) + P_k(1)).$$

Example 3.2. Let $M = \mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ and J_k the Bessel function of the first kind of order k , $k = 0, 1, 2, \dots$. Then if $\lambda_k^1 < \lambda_k^2 < \dots$ are the positive zeroes of J_k , using polar coordinates on \mathbb{D} we have

$$\mathbb{E}[hR_{(r,\theta)}^\alpha hR_{(R,\phi)}^\alpha] = \frac{\sqrt{2}}{\pi} \sum_{k,l} \frac{(\lambda_k^l)^{-(d+2\alpha)}}{|J_{k+1}(\lambda_k^l)|} J_k(\lambda_k^l r) J_k(\lambda_k^l R) (\cos(k(\theta - \phi)) + \sin(k(\theta + \phi))).$$

3.2.2. The Non-Compact Case. For the case of M non-compact, first let us show by example that we cannot establish existence in general.

Example 3.3. Let $M = \mathbb{S}^1 \times \mathbb{R}$. Then we have

$$H_t^M((\theta, x), (\phi, y)) = H_t^{\mathbb{S}}(\theta, \phi) H_t^{\mathbb{R}}(x, y)$$

where H^M is the heat kernel of M , $H^{\mathbb{S}}$ is the heat kernel of \mathbb{S}^1 , and $H^{\mathbb{R}}$ is the usual heat kernel on \mathbb{R} (see [19], Theorem 9.11).

We then have that

$$\begin{aligned} H_t^M((\theta, x), (\theta, x)) - 2H_t^M((\theta, x), (\phi, y)) + H_t^M((\phi, y), (\phi, y)) &\sim \frac{1}{\pi} \frac{1 - e^{\frac{-|x-y|^2}{4t}}}{\sqrt{(4\pi t)}} \\ &= O(t^{\frac{3}{2}}) \quad \text{as } t \rightarrow \infty \end{aligned}$$

for any $(\theta, x), (\phi, y) \in M$. So $\mathbb{E}[|R_p^\alpha|^2] = \infty \forall p \in M$ and $\alpha \geq 1/2$ and thus R^α does not exist over M for this range of α . Using \mathbb{S}^2 instead in the above we have that R^α fails to exist for all $\alpha \in (0, 1)$.

However, for certain manifolds such that $\text{Vol}(M) < \infty$ we have a situation similar to the compact case:

Theorem 3.3. *Suppose M is non-compact with $\text{Ric}(M) \geq -\kappa^2$, $\kappa \in \mathbb{R}$, and $\text{Vol}(M) < \infty$. Let $\bar{\lambda}(M) = \inf_{\Omega \subset M} \{\lambda_1 : \sigma(\Omega) = \{\lambda_k\}_{k=0}^\infty\}$ where the infimum is taken over regular domains $\Omega \subset M$ and $\sigma(\Omega)$ denotes the Dirichlet spectrum of Ω . Then if $\bar{\lambda}(M) > 0$ R^α exists over M for any $\alpha \in (0, 1)$ and hR^α does not.*

Proof. That hR^α does not exist follows from the fact that on such M

$$\lim_{t \rightarrow \infty} H_t(x, y) = \frac{1}{\text{Vol}(M)} \neq 0 \quad \forall x, y \in M.$$

For R^α , under the hypothesis of the theorem it was shown in [24] that

$$H_t(x, y) - \frac{1}{\text{Vol}(M)} = O\left(e^{-\frac{\bar{\lambda}(M)}{2}t}\right)$$

and so (3.2) converges $\forall \alpha \in (0, 1)$. □

We now turn to our main existence theorem for the Riesz fields over non-compact manifolds followed by some examples. Below we use the following notation:

$$D_p(r) \equiv \{x \in M : d(x, p) < r\}$$

and

$$V_p(r) \equiv \text{Vol}(D_p(r)) = \int_{D_p(r)} dV_g.$$

We write $H_t = \overline{O}(t^{-\frac{\nu}{2}})$ if there exist two distinct points $x_k \in M$, $k = 1, 2$, and constants $C_k > 0$ such that

$$H_t(x_k, x_k) \leq C_k t^{-\frac{\nu}{2}} \quad \forall t \geq 1.$$

In that case using Theorem 1.1 of [18] we know that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that for all $t \geq 1$ and all $x, y \in M$

$$H_t(x, y) \leq C_\delta t^{-\frac{\nu}{2}} e^{-\frac{d(x, y)^2}{(4+\delta)t}}.$$

Theorem 3.4. *Let M be non-compact.*

(1) *Suppose $\text{Ric}(M) \geq 0$. Then hR^α does not exist for any $\alpha \in (0, 1)$. If*

$$H_t = \overline{O}\left(t^{-\left(\frac{d}{2}-\beta\right)}\right)$$

and

$$\overline{\lim}_{r \rightarrow \infty} \frac{V_x(r)}{r^{d-2\beta}} < \infty \quad \forall x \in M$$

for some $\beta \in [0, 1)$ then R^α exists over M for any $\alpha \in (0, 1 - \beta)$.

(2) *Suppose that*

$$H_t = \overline{O}\left(t^{-\left(\frac{d}{2}+\beta\right)}\right)$$

for some $\beta > 0$. Then hR^α (and thus R^α also) exists for any $\alpha \in (0, \min\{\beta, 1\})$.

Proof. (1): To begin we note that our hypothesis $H_t = \overline{O}(t^{-(d/2-\beta)})$ implies the following gradient bound for H_t (see [11]): For all $x, y \in M$ and $t \geq 1$

$$(3.6) \quad |\nabla_x H_t(x, y)| \leq C'_\delta t^{-(\frac{d}{2}-\beta+\frac{1}{2})} e^{-\frac{d(x,y)^2}{(4+\delta)t}}$$

for some constant $C'_\delta > 0$.

Recall that in order for (3.2) to converge it is sufficient to show that

$$\int_1^\infty t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, o) + H_t(o, o)) dt < \infty$$

for the specified range of α . To that end let $D = D_p(r)$ be a disk containing x and o . We first apply the mean value theorem:

$$\begin{aligned} & \int_1^\infty t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, o) + H_t(o, o)) dt \\ &= \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |H_t(x, z) - H_t(o, z)|^2 dV_g(z) dt \\ &\leq d(x, o)^2 \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2 dV_g(z) dt \end{aligned}$$

for some ξ_z lying on some curve (parametrized to have unit velocity) contained in D_p and joining x and o . We now apply (3.6),

$$\begin{aligned} & \int_1^\infty t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2 dV_g(z) dt \\ &\leq C \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_M e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dV_g(z) dt. \end{aligned}$$

We have

$$\begin{aligned} & \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_M e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dV_g(z) dt \\ &= \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_D e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dV_g(z) dt \\ &\quad + \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} \int_{M \setminus D} e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dV_g(z) dt \\ &\leq \text{Vol}(D) \int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} dt \\ &\quad + \int_{M \setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dt dV_g(z). \end{aligned}$$

By hypothesis $\int_1^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} dt < \infty$ so we only need to show

$$\int_{M \setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dt dV_g(z) < \infty.$$

We have

$$\begin{aligned} & \int_{M \setminus D} \int_0^\infty t^{-\frac{d}{2}+\alpha+2\beta-2} e^{-\frac{2d(\xi_z, z)^2}{(4+\delta)t}} dt dV_g(z) \\ &= \left(\frac{4+\delta}{2} \right)^{\frac{d}{2}-\alpha-2\beta+1} \Gamma\left(\frac{d}{2}-\alpha-2\beta+1\right) \int_{M \setminus D} d(\xi_z, z)^{-d+2\alpha+4\beta-2} dV_g(z). \end{aligned}$$

Recall $D = D_p(r)$ and let

$$A_k = D_p(r+k) \setminus D_p(r+k-1) \quad k = 1, 2, 3, \dots$$

By monotone convergence

$$\begin{aligned} \int_{M \setminus D} d(\xi_z, z)^{-d+2\alpha+4\beta-2} dV_g(z) &= \sum_{k=1}^\infty \int_{A_k} d(\xi_z, z)^{-d+2\alpha+4\beta-2} dV_g(z) \\ &\leq \sum_{k=1}^\infty \frac{\text{Vol}(A_k)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\ &= \sum_{k=1}^\infty \frac{V_p(r+k) - V_p(r+k-1)}{(r+k-1)^{d-2\alpha-4\beta+2}}. \end{aligned}$$

Because $\text{Ric}(M) \geq 0$ we have (see [12] or [9])

$$V_p(cr) \leq c^d V_p(r) \quad \forall r > 0, c \geq 1.$$

Thus

$$\begin{aligned} \sum_{k=1}^\infty \frac{V_p(r+k) - V_p(r+k-1)}{(r+k-1)^{d-2\alpha-4\beta+2}} &\leq \sum_{k=1}^\infty \frac{V_p(r+k-1) \left(\frac{(r+k)^d - (r+k-1)^d}{(r+k-1)^d} \right)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\ &\leq C \sum_{k=1}^\infty \frac{(r+k-1)^{d-2\beta} \left(\frac{(r+k)^d - (r+k-1)^d}{(r+k-1)^d} \right)}{(r+k-1)^{d-2\alpha-4\beta+2}} \\ &= C \sum_{k=1}^\infty \frac{(r+k)^d - (r+k-1)^d}{(r+k-1)^{d-2\alpha-2\beta+2}} \end{aligned}$$

The convergence of this last sum is equivalent to that of

$$\sum_{k=1}^\infty \frac{k^{d-1}}{k^{d-2\alpha-2\beta+2}} = \sum_{k=1}^\infty k^{2\alpha+2\beta-3}.$$

By hypothesis $\alpha < 1 - \beta$, which implies

$$\sum_{k=1}^{\infty} k^{2\alpha+2\beta-3} < \sum_{k=1}^{\infty} k^{-(1+\epsilon)} < \infty$$

for some $\epsilon > 0$.

To see that hR^α does not exist on M for any α , we note that (see e.g. [12])

$$\text{Ric}(M) \geq 0 \Rightarrow H_t(x, y) \geq (4\pi t)^{-\frac{d}{2}} e^{-\frac{d(x, y)^2}{4t}}$$

for all $x, y \in M$ and $t > 0$. Thus

$$\int_0^\infty t^{\frac{d}{2}+\alpha-1} H_t(x, y) dt = \infty$$

for all $x, y \in M$ and any $\alpha \in (0, 1)$.

To prove (2), simply write

$$\int_1^\infty t^{\frac{d}{2}+\alpha-1} H_t(x, y) dt \leq C \int_1^\infty t^{\alpha-\beta-1} dt < \infty.$$

□

Remark 3.2. There is by now a large body of literature relating geometric or functional analytic conditions on M and the large-time decay of the heat kernel. We will not attempt to make a survey, but only mention the names of some researchers that have worked in the area recently: I. Chavel and E.A. Feldman, T. Coulhon, A. Grigor'yan, and E.B. Davies, among others. We suggest that the interested reader begin with the work of these authors and the references therein.

We are now in a position to show that, over \mathbb{R}^d , R^α agrees up to a constant with the fBf^α in distribution. We could do this abstractly using arguments along the lines of Section 3.1, however we can also make a simple explicit calculation. Note that \mathbb{R}^d satisfies the first hypothesis of Theorem 3.4 with $\beta = 0$. Thus R^α exists there and if we choose $o = 0$ has covariance

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2}+\alpha-1} (H_t(0, 0) - H_t(x, 0) - H_t(y, 0) + H_t(x, y)) dt.$$

Proposition 3.5. *If $M = \mathbb{R}^d$ then $H_t(x, y) = \frac{1}{\sqrt{(4\pi t)^d}} e^{-\frac{\|x-y\|^2}{4t}}$ and for all $x, y \in \mathbb{R}^d$ and for $\alpha \in (0, 1)$*

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = C_\alpha (\|x\|^{2\alpha} + \|y\|^{2\alpha} - \|x - y\|^{2\alpha})$$

where C_α is the positive constant given by

$$C_\alpha = \frac{-\Gamma(-\alpha)}{4^{\frac{d}{2}+\alpha} (\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2} + \alpha)}.$$

Proof. First note that if either $x = 0$ or $y = 0$ the result is trivial; thus we assume otherwise. The integral defining $\mathbb{E}[R_x^\alpha R_y^\alpha]$ reduces to

$$\frac{1}{\sqrt{(4\pi)^d}} \int_0^\infty t^{\alpha-1} \left(1 - e^{-\frac{\|x\|^2}{4t}} - e^{-\frac{\|y\|^2}{4t}} + e^{-\frac{\|x-y\|^2}{4t}}\right) dt,$$

which we recognize as a Mellin transform. Let $F_a(t) = \chi_{[a,\infty)}(t) - e^{-\frac{\|x\|^2}{4t}} - e^{-\frac{\|y\|^2}{4t}} + e^{-\frac{\|x-y\|^2}{4t}}$ with $a > 0$. Then $F_a(t) = O(t)$ as $t \rightarrow \infty$ and $F_a(t) = o(t^N)$ as $t \rightarrow 0 \forall N > 0$. Thus

$$\int_0^\infty t^{s-1} F_a(t) dt$$

converges absolutely for all $s \in \mathbb{C}$ with $\Re(s) < 1$ and defines an analytic function there.

On the other hand for $-1 < \Re(s) < 0$ we have by direct calculation that

$$\int_0^\infty t^{s-1} F_a(t) dt = \frac{a^s}{s} + \frac{-\|x\|^{2s} - \|y\|^{2s} + \|x-y\|^{2s}}{4^s} \Gamma(-s).$$

By analytic continuation this last equality holds for $0 < \Re(s) < 1$ as well. For such s we have by dominated convergence

$$\int_0^1 t^{s-1} F_0(t) dt = \lim_{a \rightarrow 0} \int_0^1 t^{s-1} F_a(t) dt.$$

Now for $a < 1$

$$\int_1^\infty t^{s-1} F_a(t) dt = \int_1^\infty t^{s-1} F_0(t) dt$$

and so, noting $F_0(t) \geq 0$, we have using dominated convergence

$$\begin{aligned} \int_0^\infty t^{s-1} F_0(t) dt &= \int_0^1 t^{s-1} F_0(t) dt + \int_1^\infty t^{s-1} F_0(t) dt \\ &= \left(\lim_{a \rightarrow 0^+} \int_0^1 t^{s-1} F_a(t) dt \right) + \int_1^\infty t^{s-1} F_0(t) dt \\ &= \lim_{a \rightarrow 0^+} \left(\int_0^1 t^{s-1} F_a(t) dt + \int_1^\infty t^{s-1} F_0(t) dt \right) \\ &= \lim_{a \rightarrow 0^+} \int_0^\infty t^{s-1} F_a(t) dt \\ &= \frac{-\|x\|^{2s} - \|y\|^{2s} + \|x-y\|^{2s}}{4^s} \Gamma(-s) \end{aligned}$$

□

Example 3.4. Suppose M is non-compact with $Ric(M) \geq 0$ and

$$\lim_{R \rightarrow \infty} \frac{V_p(R)}{R^d} = \theta \in (0, 1)$$

for some $p \in M$ (cf. the Bishop-Gromov comparison theorem). Then R^α exists over M for any $\alpha \in (0, 1)$ and hR^α does not. Indeed, in [26] it is shown that $H_t(x, y) = O(t^{-\frac{d}{2}})$ for every $x, y \in M$. Theorem 3.4 applies once we note that for all $p \in M$

$$\text{Ric}(M) \geq 0 \Rightarrow V_p(R) \leq \omega_d R^d \quad \forall R \geq 0,$$

ω_d being the volume of the unit ball in \mathbb{R}^d .

Example 3.5. If M is simply connected with all sectional curvatures $K \leq k$ for some $k < 0$ and $\text{Ric}(M) \geq -\kappa^2 > -\infty$ then hR^α exists over M for any $\alpha > 0$. For example this holds if $M = \mathbb{H}^d$, d -dimensional hyperbolic space. This follows from [27] in which it is shown that $\sigma(-\Delta) \subset [(n-1)^2 \frac{|k|}{4}, \infty)$, which in turn implies the following upper bound on H_t (see [13]):

$$H_t(x, y) \leq C e^{\frac{(n-1)^2 k t}{4}} \quad \forall t \geq 1$$

for some $C > 0$ and all $x, y \in M$. Theorem 3.4 then applies.

In particular for $M = \mathbb{H}^2$, letting $\rho = d(x, y)$ we have the well known formula

$$H_t(x, y) = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{1}{4}t} \int_{\rho}^{\infty} \frac{s e^{-\frac{s^2}{4t}}}{\cosh(s) - \cosh(\rho)} ds.$$

Then

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] = \frac{\sqrt{2}}{(4\pi)^{\frac{3}{2}} \Gamma(1+\alpha)} \int_0^\infty \int_{\rho}^{\infty} t^{\alpha-\frac{3}{2}} \frac{s e^{-\frac{1+s^2}{4t}}}{\cosh(s) - \cosh(\rho)} ds dt.$$

Remark 3.3. On negatively curved manifolds, hR^α can also be viewed as an extension of the fBf^α in the following way: In Section 3.1 we saw how the covariance of the fBf^α is the integral kernel of the operator A^*A where

$$A(f) = (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(x) - (-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}(f)(0),$$

which can be seen as a correction to $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ when this operator does not have an integral kernel. However on manifolds with spectrum as in example 3.5 $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ does have an integral kernel and no correction is needed. So if we view the fBf^α as the GRF with covariance that is the kernel of the minimal correction to $(-\Delta)^{-(\frac{d}{4}+\frac{\alpha}{2})}$ that yields an integral operator, then on such manifolds as above we obtain the hR^α .

3.3. Distributional Scaling and Invariance. Having established existence on a variety of manifolds we turn next to establishing distributional and sample path properties of the Riesz fields.

Definition 3.2. Let (M, g) be a complete Riemannian manifold and $I(M)$ the group of isometries of (M, g) . If Y_x is a centered GRF over (M, g) we say that Y_x is *stationary* (or *homogeneous*) if

$$\mathbb{E}[Y_{\iota(x)} Y_{\iota(y)}] = \mathbb{E}[Y_x Y_y]$$

for any $\iota \in I(M)$ and all $x, y \in M$. We say Y_x has *stationary (or homogeneous) increments* if

$$\mathbb{E}[|Y_{\iota(x)} - Y_{\iota(y)}|^2] = \mathbb{E}[|Y_x - Y_y|^2]$$

for any $\iota \in I(M)$ and all $x, y \in M$.

Because for any manifold (M, g) we have $H_t(\iota(x), \iota(y)) = H_t(x, y)$ for any $\iota \in I(M)$ (see [19], Theorem 9.12) it is clear from the definitions,

$$\mathbb{E}[R_x^\alpha R_y^\alpha] = \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} (H_t(x, y) - H_t(x, o) - H_t(y, o) + H_t(o, o)) dt$$

and

$$\mathbb{E}[hR_x^\alpha hR_y^\alpha] = \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} H_t(x, y) dt,$$

that when they exist, R^α and hR^α have stationary increments and are stationary respectively.

If the Riesz fields are to generalize the fractional Brownian fields they should possess a distributional scaling property analogous to that of the fractional Brownian fields over \mathbb{R}^d . In order to show this we must first define what it means to scale our manifold (M, g) by a factor $c > 0$. By scaling we mean changing the metric g on M so that $d(x, y) \mapsto cd(x, y)$ and the way to do this is to replace g by c^2g . If (M, g) is a regular domain in (N, g) then to scale M we scale N and so the following applies as well to regular domains.

Proposition 3.6. *Let (M, g) be a complete Riemannian manifold or regular domain. Both the Riesz field R^α and the stationary Riesz field hR^α over (M, g) are self-similar of order α (if they exist on M) in the sense that if \bar{R}^α and $h\bar{R}^\alpha$ are the Riesz fields over (M, c^2g) then*

$$c^\alpha R_x^\alpha \stackrel{d}{=} \bar{R}_x^\alpha$$

and

$$c^\alpha hR_x^\alpha \stackrel{d}{=} h\bar{R}_x^\alpha$$

for any $c > 0$.

This clearly agrees with the usual definition of self-similarity for a random field Y_x over \mathbb{R}^d , $c^\alpha Y_{\frac{1}{c}x} \stackrel{d}{=} Y_x$.

Proof. First we note from the coordinate expression for Δ , if we denote by Δ_g the Laplacian of (M, g) and $H_t^g(x, y)$ the corresponding heat kernel, we have $\Delta_{c^2g} = \frac{1}{c^2}\Delta_g$. But then

because $L^2(M, dV_g) = L^2(M, dV_{c^2g})$ we can write

$$\begin{aligned} \int_M c^d H_t^{c^2g}(x, y) f(y) dV_g(y) &= \int_M H_t^{c^2g}(x, y) f(y) dV_{c^2g}(y) \\ &= e^{-t\Delta_{c^2g}}(f) \\ &= e^{-\frac{t}{c^2}\Delta_g}(f) \\ &= \int_M \frac{H_t^g}{c^{\frac{d}{2}}}(x, y) f(y) dV_g(y) \end{aligned}$$

for any $f \in L^2(M, dV_g)$. Thus by symmetry

$$\frac{1}{c^d} H_{\frac{t}{c^2}}^g(x, y) = H_t^{c^2g}(x, y) \quad \forall x, y \in M.$$

We then have

$$\begin{aligned} c^{2\alpha} \mathbb{E}[R_x^\alpha R_y^\alpha] &= \frac{c^{2\alpha}}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} (H_t^g(x, y) - H_t^g(o, x) - H_t^g(o, y) + H_t^g(o, o)) dt \\ &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} \frac{1}{c^d} \left(H_{\frac{t}{c^2}}^g(x, y) - H_{\frac{t}{c^2}}^g(o, x) - H_{\frac{t}{c^2}}^g(o, y) + H_{\frac{t}{c^2}}^g(o, o) \right) dt \\ &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} \left(H_t^{c^2g}(x, y) - H_t^{c^2g}(o, x) - H_t^{c^2g}(o, y) + H_t^{c^2g}(o, o) \right) dt \\ &= \mathbb{E}[\bar{R}_x^\alpha \bar{R}_y^\alpha] \end{aligned}$$

and similarly for hR^α . □

Remark 3.4. Here we see that hR^α exhibits essentially non-Euclidean phenomena; on \mathbb{R}^d there cannot exist a GRF that is both stationary and self similar (see e.g. [3]). We will return to the questions this raises in Section 5.

3.4. Hölder Regularity. If M is any Riemannian manifold or regular domain with heat kernel $H_t(x, y)$ then the maximum principle implies

$$H_t(x, y) \leq H_t(x, x) \quad \forall x, y \in M$$

with equality if and only if $y = x$. We then have that

$$H_t(x, x) - 2H_t(x, y) + H_t(y, y) > 0 \quad \forall y \neq x.$$

In particular $\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2]$ and $\mathbb{E}[|hR_x^\alpha - hR_y^\alpha|^2]$ both define metrics on M when they exist.

Note also that

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] = \mathbb{E}[|hR_x^\alpha - hR_y^\alpha|^2]$$

when both exist. In particular in the proof below we will not distinguish these two metrics as the context of the Theorem will make clear which is being discussed.

We are now in a position to prove the following:

Theorem 3.7. *Let M be a compact Riemannian manifold, a regular domain, or non-compact under the hypothesis of Theorem 3.4. Then R^α (resp. hR^α) has a version such that with probability 1 all sample paths are Hölder continuous of order γ for any $\gamma < \alpha$ and fail to satisfy a Hölder condition on a dense subset of M for any $\gamma > \alpha$.*

Proof. In order to apply Theorem A.4 in the appendix we need to compare the metric $\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2]$ (resp. $\mathbb{E}[|hR_x^\alpha - hR_y^\alpha|^2]$) on (M, g) with the metric $d(x, y)$ derived from g , in particular study the boundedness of

$$(3.7) \quad \frac{\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2]}{(d(x, y))^{2\gamma}}$$

for $d(x, y)$ small and $\gamma \in (0, 1)$. What we will show is that this ratio is unbounded if $\gamma > \alpha$ and approaches zero if $\gamma < \alpha$.

Our approach to controlling (3.7) will be to split the integral defining $\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2]$ into two parts:

$$(3.8) \quad \int_0^\infty t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt$$

$$(3.9) \quad = \int_0^1 t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt$$

$$+ \int_1^\infty t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt.$$

We start with (3.8). Recall that in any case around any point p there is a closed disk D_p such that (2.1) holds with $\Phi \in C^k(\overline{D_p} \times \overline{D_p} \times [0, T])$ where we can choose $k > 2$ and $T > 0$.

As a consequence we have, denoting the integral (3.8) by I_1 and $d(x, y)$ by ρ ,

$$(3.10) \quad I_1 = (4\pi)^{-\frac{d}{2}} \int_0^1 t^{\alpha-1} (\Phi(t, x, x) + \Phi(t, y, y) - 2\Phi(t, x, y) e^{-\frac{\rho^2}{4t}}) dt.$$

Because $\Phi \in C^k(\overline{D_p} \times \overline{D_p} \times [0, T])$ with $k > 2$ and is symmetric, by Lemma A.1 in the appendix,

$$\Phi(t, x, x) + \Phi(t, y, y) - 2\Phi(t, x, y) = O(\rho^2) \quad \text{as } \rho \rightarrow 0$$

uniformly for $t \in [0, 1]$. Thus we have

$$\int_0^1 t^{\alpha-1} (\Phi(t, x, x) + \Phi(t, y, y) - 2\Phi(t, x, y) e^{-\frac{\rho^2}{4t}}) dt = \int_0^1 t^{\alpha-1} O(1 - e^{-\frac{\rho^2}{4t}}) dt + O(\rho^2).$$

Because

$$\int_0^1 t^{\alpha-1} (1 - e^{-\frac{\rho^2}{4t}}) dt = \rho^{2\alpha} \int_0^1 t^{\alpha-1} (1 - e^{-\frac{1}{4t}}) dt$$

and $\int_0^\infty t^{\alpha-1} (1 - e^{-\frac{1}{4t}}) dt < \infty$,

$$(3.11) \quad I_1 = O(\rho^{2\alpha}) = O(d(x, y)^{2\alpha}) \quad \text{as } d(x, y) \rightarrow 0$$

for $x, y \in D_p$.

For (3.9), which we denote I_2 , we first deal with the case of M compact. Using (2.2) we have for $t \geq 1$

$$\begin{aligned} H_t(x, x) - 2H_t(x, y) + H_t(y, y) &= \sum_{k=0}^{\infty} e^{-\lambda_k t} |\phi_k(x) - \phi_k(y)|^2 \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t} |\phi_k(x) - \phi_k(y)|^2 \\ &\leq d(x, y)^2 \sum_{k=1}^{\infty} e^{-\lambda_k t} \|\nabla \phi_k\|_{\infty}^2. \end{aligned}$$

Now we apply the following bound on $\|\nabla \phi_k\|_{\infty}$ (see [29]):

$$\|\nabla \phi_k\|_{\infty} \leq C_M \lambda_k^{\frac{d+1}{4}}$$

where C_M is a constant depending only on M . We then have

$$H_t(x, x) - 2H_t(x, y) + H_t(y, y) \leq C_M d(x, y)^2 \sum_{k=1}^{\infty} e^{-\lambda_k t} \lambda_k^{\frac{d+1}{4}} = C_M d(x, y)^2 O\left(e^{-\lambda_1 t}\right),$$

which yields

$$(3.12) \quad I_2 \leq C_M d(x, y)^2 \int_1^{\infty} t^{\frac{d}{2}+\alpha-1} O\left(e^{-\lambda_1 t}\right) dt = C d(x, y)^2$$

as $\lambda_1 > 0$.

If M is a regular domain then a similar argument using the corresponding bound (see [31])

$$\|\nabla \phi_k\|_{\infty} \leq C_M \lambda_k^{\frac{d+1}{4}}$$

for the Dirichlet eigenfunctions on M we obtain (3.13) in this case as well. Thus for either M compact or a regular domain

$$I_2 = O\left(d(x, y)^2\right) \quad \text{as } d(x, y) \rightarrow 0.$$

Turning now to the case of M non-compact, first suppose the first hypothesis of Theorem 3.4 is in force. As in that proof we have, for x, y contained in a sufficiently small geodesic disc,

$$\begin{aligned} &\int_1^{\infty} t^{\frac{d}{2}+\alpha-1} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt \\ &= \int_1^{\infty} t^{\frac{d}{2}+\alpha-1} \int_M |H_t(x, z) - H_t(y, z)|^2 dV_g(z) dt \\ &\leq d(x, y)^2 \int_1^{\infty} t^{\frac{d}{2}+\alpha-1} \int_M |\nabla_x H_t(\xi_z, z)|^2 dV_g(z) dt, \end{aligned}$$

which was shown to be finite.

Next suppose the second hypothesis holds. For this case we will use a Schauder estimate and Lemma A.1: We choose a geodesic disc D_p and let L be Δ in geodesic normal coordinates on D_p , $\mathbf{D} = \exp^{-1}(D_p)$, $P = \partial_t - L$ on $C^\infty(\mathbf{D} \times (0, 1))$, and $u(x', y', t) \in C^\infty(\mathbf{D} \times \mathbf{D} \times (0, 1))$ be $H_t(x, y)$ in our chosen coordinates. For any $T > 0$ we then have

$$Pu(x', y', t + T) = \partial_t u(x', y', t + T) - L_{x'} u(x', y', t + T) = 0$$

for each for all $x', y', t \in \mathbf{D} \times \mathbf{D} \times (0, 1/2)$. In other words, u satisfies $Pu(x', y', t) = 0$ on $\mathbf{D} \times (T, T + 1/2)$ for each $y' \in \mathbf{D}$ and $T > 0$.

Because L is uniformly elliptic on \mathbf{D} and its coefficients are all C^∞ (and independent of T, t), using the Schauder estimate (Theorem 5 p.64 in [15] and choosing $\alpha = 1$) we obtain for each closed disk \mathbf{D}_r contained in \mathbf{D} a constant $K_r > 0$ such that

$$\sup_{(x', t) \in \mathbf{D}_r \times (0, 1/2)} \left| \frac{\partial^2 u}{\partial x'_i \partial x'_j}(x', y', t + T) \right| \leq K_r \sup_{(x', t) \in \mathbf{D}_r \times (0, 1)} |u(x', y', t + T)|$$

for each i, j and $y' \in D_r$. We then have

$$\sup_{(x', y', t) \in \mathbf{D}_r \times \mathbf{D}_r \times (0, 1/2)} \left| \frac{\partial^2 u}{\partial x'_i \partial x'_j}(x', y', t + T) \right| \leq K_r \sup_{(x', y', t) \in \mathbf{D}_r \times \mathbf{D}_r \times (0, 1/2)} |u(x', y', t + T)|.$$

We note that K_r is independent of T and by our hypothesis

$$\sup_{(x, y) \in D_p \times D_p} H_t(x, y) \leq Ct^{-(\frac{d}{2} + \beta)}, \beta > 0. \text{ Thus, returning to } D_r = \exp(\mathbf{D}_r), \text{ for all } T > 1$$

$$\sup_{(x, y, t) \in D_r \times D_r \times (0, 1/2)} \left| \frac{\partial^2 H}{\partial x_i \partial x_j}(x, y, t + T) \right| \leq CK_r T^{-(\frac{d}{2} + \beta)}.$$

Then applying Lemma A.1 and assuming without loss of generality we have chosen our disc D_p such that the above estimates hold, we have

$$\begin{aligned} & \int_1^\infty t^{\frac{d}{2} + \alpha - 1} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt \\ & \leq Cd(x, y)^2 \int_1^\infty t^{\frac{d}{2} + \alpha - 1} \sup_{\overline{D_p} \times \overline{D_p}} \left| \sum_{i, j=1}^d \frac{\partial^2 H}{\partial x_i \partial x_j}(t, x, y) \right| dt \\ & \leq Cd(x, y)^2 \int_1^\infty t^{\frac{d}{2} + \alpha - 1} (t - 1/2)^{-(\frac{d}{2} + \beta)} dt \end{aligned}$$

for some $C > 0$. By hypothesis $\beta > 0$, so $\int_1^\infty t^{\frac{d}{2} + \alpha - 1} (t - 1/2)^{-(\frac{d}{2} + \beta)} dt < \infty$. Lastly recall that when hR^α exists for $\alpha \in (0, b)$ for some $b > 0$ then R^α does as well. Moreover in that case

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] = \mathbb{E}[|hR_x^\alpha - hR_y^\alpha|^2],$$

so in the second case of Theorem 3.4 the arguments above apply to R^α as well.

Thus in each case from the preceeding discussion we know that for each $p \in M$ there exists a closed disc D_p centered at p such that for all $\gamma \leq \alpha$

$$\mathbb{E}[|R_x^\alpha - R_y^\alpha|^2] \leq C_p(d(x, y)^{2\gamma})$$

for some constant $C_p > 0$ (or similarly for hR^α) and all $x, y \in D_p$ and that such a condition fails for any $\gamma > \alpha$ in light of (3.11). Then by Theorem A.4 in the appendix R^α (resp. hR^α) is almost surely uniformly Hölder continuous over D_p of order γ for any $\gamma < \alpha$. Moreover from the discussion following Theorem A.4 there is a dense subset of D_p on which R^α (resp. hR^α) fails to satisfy any Hölder condition of order $\gamma > \alpha$ with probability 1. By modifying R^α (resp. hR^α) on a set of P -measure zero we then obtain the same result for any compact $K \subset M$. Lastly by taking a compact exhaustion of M we obtain the statement of the Theorem. \square

4. THE BESSEL FIELD

We now turn to constructing stationary counterparts to R^α by analogy with the Brownian motion and Ornstein-Uhlenbeck processes on \mathbb{R} . We define the *Bessel Field* of order $\alpha \in (0, 1)$ by

$$(4.1) \quad B_x^\alpha \stackrel{d}{=} \frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} e^{-t} H_t(x, z) dt dW(z),$$

which extends the Ornstein-Uhlenbeck fields with covariance given (up to a constant) by

$$\int_{\mathbb{R}^d} \frac{e^{i\langle x, y \rangle}}{(1 + |\xi|^2)^{\frac{d}{2} + \alpha}} d\xi.$$

These fields are altogether more well behaved than the Riesz fields, which is not surprising in light of the analogy with the Riesz and Bessel potentials.

Theorem 4.1. *The Bessel field exists over any complete Riemannian manifold or regular domain M for all $\alpha \in (0, 1)$.*

Proof. Proceeding as for hR^α , for each $x, y \in M$

$$\begin{aligned} \mathbb{E}[B_x^\alpha B_y^\alpha] &= \left(\frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \right)^2 \int_M \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} e^{-t} H_t(x, z) dt \int_0^\infty s^{\frac{d}{4} + \frac{\alpha}{2} - 1} e^{-s} H_s(y, z) ds dV_g(z) \\ &= \left(\frac{1}{\Gamma\left(\frac{d}{4} + \frac{\alpha}{2}\right)} \right)^2 \int_0^\infty \int_0^\infty t^{\frac{d}{4} + \frac{\alpha}{2} - 1} s^{\frac{d}{4} + \frac{\alpha}{2} - 1} e^{-(t+s)} H_{t+s}(x, y) dt ds \\ (4.2) \quad &= \frac{1}{\Gamma\left(\frac{d}{2} + \alpha\right)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} e^{-t} H_t(x, y) dt \end{aligned}$$

From the fact that the heat kernel always satisfies $\overline{\lim}_{t \rightarrow \infty} H_t(x, y) < \infty$ for any x and y , we see that (4.2) converges everywhere on $M \times M$. \square

Clearly B_x^α is stationary and we can see that it does not possess the scaling properties of the Riesz fields. Turning to sample path regularity we have the following result.

Theorem 4.2. *The Bessel field B^α has a version with sample paths almost surely uniformly Hölder continuous of order γ for any $\gamma < \alpha$ and almost surely failing to satisfy a Hölder condition of order γ for any $\gamma > \alpha$ on a dense subset of M .*

Proof. Split the integral

$$\begin{aligned} \mathbb{E}[|B_x^\alpha - B_y^\alpha|^2] &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} \int_0^\infty t^{\frac{d}{2} + \alpha - 1} e^{-t} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt \\ &= \frac{1}{\Gamma(\frac{d}{2} + \alpha)} (I_1 + I_2) \end{aligned}$$

where

$$I_1 = \int_0^1 t^{\frac{d}{2} + \alpha - 1} e^{-t} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt$$

and

$$I_2 = \int_1^\infty t^{\frac{d}{2} + \alpha - 1} e^{-t} (H_t(x, x) - 2H_t(x, y) + H_t(y, y)) dt$$

and argue as in Theorem 3.7. □

5. CONCLUSION AND FURTHER WORK

Using a spectral theoretic approach we have constructed analogues of the fractional Brownian fields over arbitrary compact manifolds and a wide class of non-compact manifolds. However we did not establish uniqueness and, at least for $\alpha \in (0, 1/2)$, it seems unclear what such a result would look like in light of the fact that there now exist two distinct GRF's over \mathbb{S}^d and \mathbb{H}^d that have stationary increments and are self-similar. Thus one could ask how many different such fields there are over any given manifold.

We also saw in example 3.3 that R^α does not exist on $\mathbb{S}^1 \times \mathbb{R}$ for $\alpha > 1/2$. This raises the following question: Does there exist any Gaussian field over $\mathbb{S}^1 \times \mathbb{R}$ with stationary increments that is also self similar of order α for some $\alpha \in (1/2, 1)$? More generally, are there geometric conditions that ensure a given manifold can have such a field defined over it?

There is one aspect of this theory we did not touch upon, that being the behavior of our fields when restricted to geodesics. For example one could require that a field when restricted to infinite geodesics became a fractional Brownian motion. However this would require an infinite geodesic and a family of isometries of the whole manifold that restrict to translation of the given geodesic, which is a strong condition indeed.

We also mentioned above that the existence of hR^α raises interesting questions regarding negatively curved manifolds and what we could loosely call hyperbolic Gaussian random fields. For example, although the proof of existence of hR^α over \mathbb{H}^d is spectral theoretic, one can ask if there are more geometric or topological conditions one can put on a manifold

M to ensure the existence of some self-similar and stationary GRF. Conversely one can ask what are the implications of such a field existing over M . Is hR^α the only such field or are there others?

One can view a GRF over a manifold as a kind of “randomization” of the manifold. One can then roughly summarize the above questions in the following way: Given a manifold M , in what ways can we randomize M and how does the answer depend on the geometry of M ?

The above is only a first attempt to state some questions at the intersection of geometry and probability that, at least on the face of it, seem novel and interesting; doubtless there are others. The study of random fields over manifolds, although its history is not short, seems to the author to still be wide open. It is our hope that the work here and the questions raised above will be of interest to both geometers and probabilists and lead to further interaction between the two.

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APPENDIX A.

First we record the following Lemma involving Taylor approximation.

Lemma A.1. *Let M be complete and suppose $f \in C^\infty(M \times M)$ is symmetric. Around any point $p \in M$ there exists a closed geodesic disk D_p centered at p and a constant $C_p > 0$ such that*

$$|f(x, x) - 2f(x, y) + f(y, y)| \leq C_p d(x, y)^2 \sup_{D_p \times D_p} \left| \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

for all $x, y \in D_p$.

Proof. Let $F \in C^2(\mathbb{R}^d)$ and recall Taylor’s Theorem: for each $p \in \mathbb{R}^d$ and all $x \in \mathbb{R}^d$

$$\begin{aligned} F(x) &= F(p) + \sum_{i=1}^d \frac{\partial F}{\partial x_i}(p)(x_i - p_i) \\ &\quad + \sum_{i,j=1}^d (x_i - p_i)(x_j - p_j) \frac{2}{1 + \delta_{ij}} \int_0^1 (1-t) \frac{\partial^2 F}{\partial x_i \partial x_j}(p + t(x - p)) dt. \end{aligned}$$

Now let $f \in C^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $f(x, y) = f(y, x)$. Fix $x, y \in \mathbb{R}^d$. Then letting $p = (x, y)$, from the symmetry of f we have

$$\begin{aligned}
& f(x, x) - 2f(x, y) + f(y, y) \\
&= \sum_{i,j=1}^d (x_i - y_i)(x_j - y_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y-x), x) dt \\
&\quad + \sum_{i,j=1}^d (x_i - y_i)(x_j - y_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(y + t(x-y), y) dt \\
&= \int_0^1 \sum_{i,j=1}^d (x_i - y_i)(x_j - y_j) (1-t) \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x + t(y-x), x) \right. \\
&\quad \left. + \frac{\partial^2 f}{\partial x_i \partial x_j}(y + t(x-y), y) \right) dt \\
&= c \sum_{i,j=1}^d (x_i - y_i)(x_j - y_j) \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x + \theta_1, x) + \frac{\partial^2 f}{\partial x_i \partial x_j}(y + \theta_2, y) \right)
\end{aligned}$$

for some constant $c > 0$ and $\theta_k \in \mathbb{R}^d$ with $\|\theta_k\|_{\mathbb{R}^d} < \|x - y\|_{\mathbb{R}^d}$. In particular for x, y in a closed disk D_ϵ of radius $\epsilon > 0$ we have

$$|f(x, x) - 2f(x, y) + f(y, y)| \leq C_1 \|x - y\|_{\mathbb{R}^d}^2 \sup_{D_\epsilon \times D_\epsilon} \left| \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

for some $C_1 > 0$.

Now suppose $f \in C^\infty(M \times M)$ is symmetric and let D_p be a geodesic disk centered at $p \in M$. Then the above implies

$$(A.1) \quad |f(x, x) - 2f(x, y) + f(y, y)| \leq C_2 d(x, y)^2 \sup_{D_p \times D_p} \left| \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j} \right|$$

for all $x, y \in D_p$.

□

A.1. Continuity of Gaussian random fields. Here we provide analogues of results given for Gaussian fields over \mathbb{R}^d in the setting of manifolds. These proofs are simple modifications of the originals and we include them for convenience. The first result is an analytical lemma, given for hypercubes in \mathbb{R}^d . We will replace the cubes with metric disks and \mathbb{R}^d by a d -dimensional manifold M . Let p be even and continuous on $[-1, 1]$, $p(|x|)$ monotone increasing, and satisfy $\lim_{x \rightarrow 0} p(x) = 0$.

Lemma A.2. (*Manifold version of Lemma 1 in [17]*): Let $f \in C(I_0)$ where $I_0 \subset M$ is a closed metric disk $I_0 = \{y \in M : d(x, y) < r\}$ contained in some normal neighborhood

about $x \in M$. Suppose that

$$\int_D \int_D \exp \left(\frac{f(x) - f(y)}{p(\text{diam}(D))} \right)^2 dx dy \leq B$$

for all closed metric disks $D \subset I_0$. Then for some $C > 0$

$$|f(x) - f(y)| \leq 8 \int_0^{d(x,y)} \sqrt{\log(BCu^{-2d})} dp(u)$$

for all $x, y \in I_0$.

Proof. Fix $x, y \in I_0$ and let D_0 be the smallest metric disk containing both (so that $\text{diam}(D_0) = d(x, y)$). Then choose a sequence of disks D_k that shrink to x and such that if $d_k = \text{diam}(D_k)$ we have

$$p(d_k) = \frac{1}{2}p(d_{k-1}).$$

Let $f_{D_k} = \text{Vol}(D_k)^{-1} \int_{D_k} f dV$. We apply Jensen's inequality to obtain

$$\begin{aligned} & \exp \left(\frac{f_{D_k} - f_{D_{k-1}}}{p(d_{k-1})} \right)^2 \\ & \leq [\text{Vol}(D_k)\text{Vol}(D_{k-1})]^{-1} \int_{D_k} \int_{D_{k-1}} \exp \left(\frac{f(x) - f(y)}{p(d_{k-1})} \right)^2 dV(x) dV(y) \\ & \leq B[\text{Vol}(D_k)\text{Vol}(D_{k-1})]^{-1}. \end{aligned}$$

We then have

$$(A.2) \quad |f_{D_k} - f_{D_{k-1}}| \leq p(d_{k-1}) \sqrt{\log(B[\text{Vol}(D_k)\text{Vol}(D_{k-1})]^{-1})}$$

By the definition of D_k we have

$$p(d_{k-1}) = 4[p(d_k) - p(d_{k+1})]$$

and by hypothesis on I_0 , $\exists C > 0$ such that

$$\text{Vol}(D_k) \geq C(d_k)^d$$

so that $d_{k+1} \leq u \leq d_k \Rightarrow u^{-2d} \leq C[\text{Vol}(D_k)\text{Vol}(D_{k-1})]^{-1}$. Then we can write (4.1) as

$$|f_{D_k} - f_{D_{k-1}}| \leq 4 \int_{d_{k+1}}^{d_k} \sqrt{\log(BCu^{-2d})} dp(u).$$

Summing these and using continuity of f we get

$$|f(x) - f_{D_0}| = \overline{\lim}_{k \rightarrow \infty} |f_{D_k} - f_{D_0}| \leq 4 \int_0^{d_1} \sqrt{\log(BCu^{-2d})} dp(u).$$

Now $d_1 < d(x, y)$ so if we need to we can replace B by a larger bound to ensure the integrand is defined, and after doing so we have

$$|f(x) - f_{D_0}| \leq 4 \int_0^{d(x,y)} \sqrt{\log(BCu^{-2d})} dp(u).$$

The argument is symmetric in x and y , so an application of the triangle inequality yields the conclusion. \square

Suppose now we are given a (centered) Gaussian random field X_x over (M, g) and consider its restriction to a closed disk I_0 as above. Suppose further that the function $K(x, y) = \mathbb{E}[X_x X_y]$ is continuous on $I_0 \times I_0$. Then $K(x, y)$ determines a positive trace class integral operator on $L^2(I_0, dV_g)$ and by Mercer's theorem we have

$$K(x, y) = \sum_{k=0}^{\infty} \lambda_k \phi_k(x) \phi_k(y)$$

uniformly on $I_0 \times I_0$, where λ_k and ϕ_k are the eigenvalues and eigenfunctions of K respectively.

Let $p(u) = \sup\{\sqrt{\mathbb{E}[|X_x - X_y|^2]} : d(x, y) \leq |u|\}$ and $X_x^n = \sum_{k=0}^n \sqrt{\lambda_k} \phi_k(x) \theta_k$ where the θ_k are independent standard normal random variables.

We then have the following adaptation of Garsia's theorem to the manifold setting:

Theorem A.3. (*Manifold version of Theorem 1 in [17]*): Suppose that for $x, y \in I_0$ as above

$$\int_0^{\text{diam}(I_0)} \sqrt{-\log(u)} dp(u) < \infty.$$

Then with probability 1

$$|X_x^m - X_y^m| \leq C \left(\sqrt{\log(B)} p(d(x, y)) + \int_0^{d(x, y)} \sqrt{-\log(u)} dp(u) \right)$$

where $C > 0$ and

$$B = \sup_m \int_{I_0} \int_{I_0} \exp \frac{1}{4} \left(\frac{X_x^m - X_y^m}{p(d(x, y))} \right)^2 dV(x) dV(y)$$

is almost surely finite. In particular the partial sums X_x^m are almost-surely equicontinuous and uniformly convergent on I_0 .

Proof. Let

$$P_n = \exp \frac{1}{8} \left(\frac{X_x^n - X_y^n}{p(d(x, y))} \right)^2 = P_{n-1} \exp \frac{1}{8} \left(\frac{(Y^n(x, y))^2 - 2Y^n(x, y)(X_x^{n-1} - X^{n-1}(y))}{p(d(x, y))} \right)^2$$

where $Y^k(x, y) = \sqrt{\lambda_k}(\phi_k(x) - \phi_k(y))\theta_k$. Then by independence of the θ_k and Jensen's inequality for conditional expectation

$$\begin{aligned}
& \mathbb{E}[P_{n+1} | P_n, \dots, P_1] \\
&= P_n \left(\mathbb{E} \left[\exp \frac{1}{8} \left(\frac{X_x^{n+1} - X_y^{n+1}}{p(d(x, y))} \right)^2 \middle| P_n, \dots, P_1 \right] \right) \\
&\geq P_n \exp \frac{1}{8} \left(\mathbb{E} \left[\left(\frac{(Y^{n+1}(x, y))^2 - 2Y^{n+1}(x, y)(X_x^{n-1} - X^{n-1}(y))}{p(d(x, y))} \right) \middle| P_n, \dots, P_1 \right] \right)^2 \\
&= P_n \exp \frac{1}{8} \left(\mathbb{E} \left[\left(\frac{(Y^{n+1}(x, y))^2}{p(d(x, y))} \right) \middle| P_n, \dots, P_1 \right] \right)^2 \\
&\geq P_n \quad a.s.
\end{aligned}$$

Thus $\{P_n\}$ is a submartingale. Next note that $\mathbb{E}[P_n^2] \leq \sqrt{2}$, as

$$\frac{X_x^n - X_y^n}{p(d(x, y))}$$

is centered, Gaussian, and has variance less than or equal to one. Then applying the classical submartingale inequalities we have

$$\mathbb{E}[\max_{m \leq n} P_m^2] \leq 4\mathbb{E}[P_n^2] \leq 4\sqrt{2}.$$

Applying the Fubini-Tonelli theorem we then have

$$\mathbb{E} \left(\int_{I_0} \int_{I_0} \max_{m \leq n} \exp \frac{1}{4} \left(\frac{X_x^n - X_y^n}{p(d(x, y))} \right)^2 dV(x) dV(y) \right) \leq 4\sqrt{2} (V(I_0))^2.$$

Letting n tend to infinity and applying monotone convergence yields

$$\mathbb{E}[B] \leq 4\sqrt{2} (V(I_0))^2 < \infty.$$

We then have that almost surely

$$\int_{I_0} \int_{I_0} \exp \frac{1}{4} \left(\frac{X_x^n - X_y^n}{p(d(x, y))} \right)^2 dV(x) dV(y) \leq B < \infty \quad \forall n$$

so that Lemma A.2 applies.

Lastly note that from

$$\mathbb{E} \left[\sum_{k=0}^{\infty} \lambda_k \theta_k^2 \right] = \sum_{k=0}^{\infty} \lambda_k = \int_{I_0} K(x, x) dV(x) < \infty$$

we obtain with probability one

$$\sum_{k=0}^{\infty} \lambda_k \theta_k^2 < \infty,$$

which together with the conclusion of Lemma A.2 implies the almost sure uniform convergence of $\{X_x^n\}$ on I_0 .

□

As remarked in [17] this result gives a sufficient condition for the existence of an almost surely continuous version of X_x . The next result establishes Hölder continuity.

Theorem A.4. (*Manifold version of Thm 8.3.2 in [1]*): *Let the field X over $I_0 \subset M$ be as above and let $\gamma = \sup\{\beta : \mathbb{E}[|X_x - X_y|^2] = o(d(x, y)^{2\beta}) \text{ uniformly on } I_0\}$. Then there exists a version of X with sample paths that are almost surely uniformly Hölder continuous over I_0 of any order $\beta < \gamma$.*

Proof. Let $\rho = d(x, y)$. First note that, with $p(u)$ as above, we have for any $L > 0$

$$\int_L^\infty p(e^{-x^2}) dx \leq c_\epsilon \int_L^\infty e^{-(\gamma-\epsilon)x^2} dx < \infty$$

for any $0 < \epsilon < \gamma$ and some constant c_ϵ . But this is equivalent to

$$\int_0^{\text{diam}(I_0)} \sqrt{-\log(u)} dp(u) < \infty.$$

Thus by the previous result we have a version (which we also denote by X) for which

$$|X_x - X_y| \leq Bp(\rho) + C \int_0^\rho \sqrt{-\log(u)} dp(u) \quad a.s.$$

for some constant $C > 0$ and some positive random variable B almost surely finite.

Now for any $0 < \epsilon < \gamma$ we have some constant $C_\epsilon > 0$ such that $p(\rho) < C_\epsilon \rho^{\gamma-\epsilon}$, and similarly $\int_0^\rho \sqrt{-\log(u)} dp(u) < C'_\epsilon \rho^{\gamma-\epsilon}$ for some $C'_\epsilon > 0$. Thus, with probability 1, for each $\epsilon > 0$ there is an almost surely finite positive random variable A_ϵ such that

$$|X_x - X_y| \leq A_\epsilon d(x, y)^{\gamma-\epsilon} \quad \forall x, y \in I_0.$$

□

Note that we can also show under the hypotheses of the theorem that in any disk of positive radius in I_0 the sample paths of X fail to be uniformly Hölder of any order greater than γ . Indeed,

$$\frac{X_x - X_y}{d(x, y)^{\gamma+\epsilon}}$$

is a centered Gaussian random variable with variance $O(d(x, y)^{-\frac{\epsilon}{2}})$ and thus becomes almost surely unbounded as $x \rightarrow y$. For example we can pick any countable dense subset of I_0 and modify X on a set of measure zero to obtain the failure of Hölder continuity at each point in the set. Any stronger converse statement will require more refined tools, i.e., local times, which we will not attempt to develop here.

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